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A MONTE CARLO APPROACH TO THE FACTORIAL INVARIANCE
PROBLEM USING THE ORTHOGONAL
PROCRUSTES SOLUTION

BY



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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled, "A Monte Carlo Approach to the Factorial Invariance Problem Using the Orthogonal Procrustes Solution," submitted by Ernest N. Skakun in partial fulfillment of the requirements for the degree of Master of Education.

ABSTRACT

Determining the extent to which factors obtained from two different studies are similar has plagued researchers for some time. The degree of factorial similarity was usually assessed by the use of various indices (Tucker, 1951; Kaiser, 1960; Wrigley and Neuhaus, 1955; and Harman 1967) or by the magnitude of the angles through which one factorial structure would have to be rotated to approximate another structure. Regardless of the technique employed for assessment, neither provided an adequate solution to the problem of factorial invariance.

Using a Monte Carlo approach, the present study attempted to develop an empirical sampling distribution from one type of factor match procedure, namely the Orthogonal Procrustes.

Twenty-two different component structure matrices (A) ranging in order from five variables by three components to twenty variables by six components served as population A matrices. One randomly generated component score matrix (F) of order one thousand observations by twenty variables provided the sample source in the study. Employing the Orthogonal Procrustes factor match procedure and a constant sample size of fifty, one thousand matches were performed for each A matrix. The average trace of $(E'E)$ and the "largest absolute value" of the error matrix were computed. Following the calculation of the frequency distributions of average trace of $(E'E)$ where E is defined as the difference between the rotated source and the target matrices and the "largest



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absolute value" for the one thousand matches, the 25, 50, 75, 90, 95, and 99 percentile points were calculated.

For several of the A matrices, the procedure was repeated using sample sizes of one hundred and one hundred and fifty. In such cases, only five hundred matches were performed.

The values for the various percentiles are presented and can be used as a guideline in evaluating the significance of the results obtained from matching two data sets when the Orthogonal Procrustes factor match is used.

In addition, cumulative frequency polygons and frequency polygons of the average trace of $(E'E)$ values were plotted for various matrices to give an indication of the shape and distribution of the graph.

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CHAPTER I

INTRODUCTION AND STATEMENT OF PROBLEM

Factor analysis is a method for extracting information from a data set. One begins with a set of observations on a number of variables and then by analyzing the intercorrelations of the variables, tries to express the measures in terms of a smaller number of reference variables. These reference variables have been called factors or components.

The term "factor analysis" is often used ambiguously. It is used generically to refer to the representation of a variable in terms of several underlying variables, and it is used specifically to refer to an analytic procedure which best reproduces the observed correlations. The latter use of the term is in contrast to "component analysis" whose purpose is to extract maximum variance. The present study is concerned specifically with component analysis. Where the term factor is used, it is used in its generic sense.

One of the most important problems that has yet to be solved by psychometricians is the problem of determining the extent of comparability of factors obtained from two different sets of data. Based upon the nature of the data, four cases may be distinguished depending upon whether the same or different variables and whether the same or different individuals have been observed in the two data sets.

The most common situation and the only one of interest in

the present investigation is the situation in which the same variables and different individuals form the basis of the analysis. Specifically the problem is one of deciding whether a component found in one sample can be identified with that found in another sample. Methods of solution for this case have been developed by Burt (1948), Tucker (1951), Ahmavaara (1954), Wrigley and Neuhaus (1955), Kaiser (1960), and Schönemann (1966).

In the early exploratory studies, generally rough methods of inspection and personal impressions sufficed as the basis for identification of similar factors in separate studies. In 1945, Zachert and Friedman (1953) administered the 23 variable Aircrew Classification Battery to 8574 aviation trainees. Two years later, the same battery was administered to 1511 pilot trainees. They then set out to compare the factorial content of the battery on war-time and post-war samples. The factor analyses of the two batteries showed that the factorial contents for both samples were similar. They used as a criterion of stability, loadings for variables that were at least .30 in both samples.

Burt (1948) in his study of emotional traits of children proposed the "proportionality criterion" for assessing the amount of agreement between factors derived from different factorial analyses. A set of factor coefficients of twelve temperamental traits for a general emotionality factor was compared with a teacher's set of independent gradings for general emotionality. A value of .931 was obtained.

Tucker (1951) analyzed two studies - one involving 18

variables for a sample of Naval Recruits and the other involving 44 variables for a sample of Aircrew and Soldiers. Ten variables were common to both samples and six factors of the smaller study were matched with six of the twelve factors of the larger. Coefficients of congruence as proposed by the same author were calculated for the six factors and those ranging from .999984 down to .939811 were accepted as defining congruent or similar factors. A value of .459717 was rejected as being too low to warrant congruence of factors.

Wrigley and Neuhaus (1955) presented the same formula as Tucker had for measuring the degree of factorial similarity.

More recently, Quereshi (1967) investigated the invariance of certain ability factors with respect to consistency of factors for a fixed set of variables and different samples. The data represented the performance of 700 children, comprising seven independent samples of 100 subjects each on the Illinois Test of Psycholinguistic Abilities and the Stanford-Binet. Coefficients of congruence as defined by Burt (1948), Tucker (1951), and Wrigley and Neuhaus (1955) were calculated across the seven samples on four factors. The author concluded that since 78 or 84 coefficients were close to unity, it would be justifiable to say that four factors possessed a high degree of stability across the samples.

Ahmavaara (1957) applied his method of transformation analysis which he proposed in 1954 to a comparison of factors as revealed by factorial analysis of ability. The degree of factorial similarity was attained by plotting the transformed matrix against

the target matrix and observing whether the points fall along a line $y = x$. The author also suggested that the average sum of squares for the difference between the transformed and target matrix may also be computed.

Hereford (1968) employed a semantic differential test to determine the component structure of the responses made by three categories of listeners to different kinds of music. She then employed Ahmavaara's technique to assess the degree of similarity between factors for the different groups. Judgments were made based on the size of the values in the transformation matrix. The writer stated that values of .9 or greater indicated a strong correspondence between factors while values below .8 indicated a lower correspondence between the factors.

Kaiser (1960) related Thurstone's four rotated primary factors obtained from anthropometric measurements on 100 adult North Ireland males with two orthogonal varimax factors based on 14 measurements on 50 London University male students. Six variables were common to both samples. The "quality of fit" of the factors between the two samples was ascertained by the mean cosine index and a value of .85 was obtained. Kaiser concluded that the value of .85 suggested that the pairing of the six variables was appropriate and that the relationship confirmed the hypothesis of similar factors in both studies.

Stewin (1969) employed Kaiser's method in matching two samples which were exposed to a battery purporting to measure conceptual system functioning. Rather than computing a mean

cosine index, Stewin based his conclusions upon the size of the cosine values in the transformation matrix. Values of .800 or greater were accepted as significant and indicated that factors were similar in both samples.

In a study intended to examine the relationship between environmental variables and different mental abilities, Mosychuk (1969) administered the Wechsler Intelligence Scale for Children (WISC) to 100 ten year old boys and interviewed the mothers with the Differential Environmental Process Variables (DEPVAR) Interview Schedule. The sample of 100 was then divided into two groups of 50 and analyses were conducted on each group. Responses on the WISC and DEPVAR were factor analyzed. Group two results were matched with the target Group One using Kaiser's procedure (1960). Values in the transformation matrix of .866 and greater were accepted as indicative of favorable matching.

The Orthogonal Procrustes solution is another method which can be used for rotating a source matrix of factor loadings to a target matrix. Such a solution was derived by Schönemann (1966).

Two samples of Illinois high school teachers were employed to rate two sets of fifteen educational objectives on thirty scales in a study conducted by Maguire (1967). Six components based on the factoring of the intercorrelations among scales were obtained. Schönemann's Orthogonal Procrustes solution was used to rotate the structure of sample two to that of sample one. Using the criterion that each row or column of the transformation matrix should contain one element near plus or minus one and the rest near zero, the

author concluded that four of the six components displayed stability across different sets of teachers and different sets of objectives.

In 1967, Taylor and Maguire tried to determine what differences existed among perceptions of broad objectives of science curriculum by three groups of people. A semantic differential consisting of eighteen concepts and twenty-seven scales was rated by three groups of people labelled as teachers, writers, and experts. Component structures were obtained for each group and the fit of pairs of component structures was determined using Schonemann's procedure. The authors concluded that no major differences existed between the groups' perceptions of science objectives. The conclusion was based on the fact that values for the solution of the Procrustes problem were close to unity.

In many of the studies described above, degree of factorial similarity has been assessed using the magnitude of the angles through which one structure would have to be rotated in order to more closely match a second structure. However the structures themselves are generally rotations of principal axes solution and therefore they are in one sense arbitrary. The "goodness" of a rotation is most often a configural judgment. That is, "goodness" is measured by result, not procedure. Therefore if two structures resulting from say principal components - varimax analyses of data for two groups, do not resemble each other initially, it may be possible to rotate one into close approximation of the other. In that case, even though the transformation matrix indicated substantial rotation was necessary, the researcher should conclude that the factor structures are similar.

In summary then, a number of approaches have been developed for assessing the degree of factorial similarity. Whenever indices such as proposed by Burt, Tucker, Wrigley and Neuhaus, and Kaiser are employed, no basis is given for determining the minimum value for the coefficient in order to define congruent factors. If one resorts to Ahmavaara's method of plotting the transformed matrix against the target matrix, then the problem of what constitutes a significant plot is encountered. If average sum of squares of the difference matrix are computed, the researcher has no idea as to the magnitude of these values. Researchers who have relied on the values in the transformation matrix as a basis for assessing structural stability have not reached agreement as to what value determines whether factors are stable across samples or variables.

Although a number of solutions have been offered for factor matching and testing whether matches are significant, the question of what constitutes a satisfactory match still prevails. Rather than using the transformation matrix of direction cosines as a guideline in assessing the significance of the matches, the present study shifts the emphasis to the difference obtained between the rotated and the target matrices. Schönemann adopted the same emphasis and in order to bring about a more efficient match, set the criterion that the difference between the rotated and the target matrix be a minimum. Thus given a source matrix A_1 and a target matrix A_2 , an orthogonal transformation matrix T is applied to A_1 to rotate it as close as possible to A_2 . In matrix form this can be expressed as

$$A_1 T = A_2 + E$$

where E is the matrix of differences between the rotated A_1 and the target matrix A_2 . By least-squares principles, E should be a minimum. Schönemann's solution requires the derivation of a unique transformation matrix in order that the criterion be fulfilled.

However, even if one resorts to using Schönemann's procedure, the problem of whether the factors in the two samples are similar still exists. This is so because no known attempt has been made at assessing the sampling distributions of goodness of fit statistics as put forth by the various methods for assessing factorial similarity.

Therefore, if a match is performed, it would be worthwhile to know whether or not the two factor solutions are significantly different. A researcher would then be enabled to conclude with certain confidence that the two factor solutions are derived from subjects from the same population.

SUMMARY

The comparison of one set of factor loadings to another set has been isolated as one of the problems facing researchers. Studies employing various techniques for matching factors have been reviewed and the inconsistency in deciding whether one set of factors matches another was demonstrated.

PURPOSE OF THE STUDY

The purpose of the present study was to develop an empirical sampling distribution from one type of factor match

procedure and to extend the principles of inferential hypothesis testing into the factor analysis area. Components were matched according to the procedure developed by Schönemann. For each match average $\text{tr}(E'E)$ and the "largest absolute value" of the error matrix were calculated. Based on the null hypothesis that two samples are drawn at random from a population exhibiting a particular component structure, an attempt was made to develop a distribution of the average $\text{tr}(E'E)$ of the error matrix. The 25, 50, 75, 90, 95, and 99 percentile points were computed for the various distributions in an attempt to provide some basis for assessing satisfactory matching.

CHAPTER II

SOME RELATED LITERATURE

The present chapter covers three related aspects of the literature: a brief history of factor analysis, the component analysis model, and factorial invariance. The latter is dealt with in more detail than the former two.

A BRIEF HISTORY OF FACTOR ANALYSIS

Although factor analysis has been used extensively in psychology, it is not a psychological theory but rather a branch of statistics. Harman (1967) has credited Spearman with the primary development of factor analysis.

Horst (1965) summarized Spearman's contribution. At the turn of the 1900's considerable debate arose as to the presence of general and special abilities in the cognitive domain. Spearman hypothesized that intelligent behavior or any cognitive variable was composed of a general ability factor and a factor unique to each variable. Several years later, Spearman applied correlational procedures to data in an attempt to verify his hypothesis of the two-factor theory of intelligence.

It was soon realized that the two-factor theory was not adequate to describe a battery of psychological measures. For instance some tests were found to have a factor in common, in addition to the general factor, that was not common to other tests. To compensate for this inadequacy, Holzinger (Harman, 1967) modified

the two-factor theory. Holzinger's bi-factor method obtained a general and one or more group factors. A variable or test was described in terms of a general factor, a group factor, and a unique factor.

Thurstone (1947) extended the work of Spearman and like Holzinger, favored a theory of many common factors. As a consequence, Thurstone's method of multiple factor analysis was popularized. For Thurstone, factor analysis had two different tasks. A factor analytic study should condense the test scores by expressing them in terms of a relatively small number of linearly independent factors or it could discover the underlying processes which produce the test performances and describe the individual differences in terms of the underlying processes. At the same time if the factor model was to express the underlying processes, Thurstone stated a number of requirements to be fulfilled by the model. Firstly, the model should be parsimonious, that is, it should use as small a number of concepts with as great a scope as possible. Secondly, the factors should be invariant. This means that factors should not be dependent on just one grouping of variables and individuals that are being analyzed at that moment, but should appear in analyses under different experimental circumstances. Lastly, the model should display uniqueness of results. One configuration of factors should occur over repeated investigations with different test batteries within the same domain.

In summary then, "The principal concern of factor analysis is the resolution of a set of variables linearly in terms of (usually) a

small number of categories or 'factors'. This resolution can be accomplished by the analysis of the correlations among the variables. A satisfactory solution will yield factors which convey all the essential information of the original set of variables." (Harman, 1967, p. 4).

B. THE COMPONENT ANALYSIS MODEL

The aim of factor analysis is to express a score on a test or variable in terms of several underlying factors. The simplest mathematical model for describing a variable in terms of several others is linear. However, Harman (1967) indicated that a researcher has several alternatives available within the linear framework. An analysis could be performed which would best reproduce the correlations between the variables or one which would extract the maximum variance from the variables. The latter alternative formulates the basis for this study. In 1933, Hotelling developed the method of principal component analysis. Algebraically this model can be represented as:

$$Z_{ji} = a_{j1}F_{1i} + a_{j2}F_{2i} + a_{j3}F_{3i} + \dots + a_{jr}F_{ri}$$

where

Z_{ji} - standard score for person i on test j

a_{jr} - component loading of test j on component r

F_{ri} - component score for person i for component r

The observed variables (Z_{ji}) are now described linearly in terms of components (F_{ri}). Successive components made a maximum contribution to the sum of the variances of the variables that are residual

after preceding components have been extracted.

Since the present study is concerned only with orthogonal solutions and the extraction of "r" components, a parsimonious expression of the above model could be represented in matrix notation as:

$$Z = A F$$

where

Z = standard score matrix of order $(n \times N)$

A = a matrix of component loadings of order $(n \times r)$

F = a matrix of component scores of order $(r \times N)$

and

N = number of observations

n = number of variables

r = number of components

Computationally, analysis proceeds from the correlation matrix R , $(n \times n)$ rather than the observed variables. Thus if $Z = AF$ is post-multiplied by its transpose and then divided by N , the number of observations, the correlation matrix R is formed.

$$Z = A F$$

$$\frac{Z Z'}{N} = \frac{A F F' A'}{N}$$

If the component scores are scaled with zero mean and unit variance and are orthogonal, then $\frac{F F'}{N} = I$, the identity matrix, and the expression reduces to $R = A A'$. The matrix A is then determined from a principal component analysis of R .

TRANSFORMATIONS

Since no unique mathematical solution exists for the unique definition of A, transformations are usually carried out on A in an attempt to meet the criterion of simple structure as defined by Thurstone (1947). Thus given a component matrix A, it may be rotated by postmultiplication by a transformation matrix T to form B. Hence

$$B = A T,$$

where $T'T = TT' = I$.

The matrix of loadings B should now meet the condition of simple structure. The transformation matrix T may be determined graphically or analytically. With the advent of computers, T has generally been determined by recourse to analytic methods such as varimax (Kaiser, 1958) and quartimax (Neuhaus and Wrigley, 1954).

Further problems are introduced when oblique rather than orthogonal transformations of the axes are made. The present study concerns only orthogonal solutions and the problems of oblique transformation need not concern us here.

C. FACTORIAL INVARIANCE

THE CONCEPT OF INVARIANCE

Thurstone (1947) specified that in factor analytic work, invariance of results is one of the requirements. Young and Householder (1940, p. 53) noted that "any consistent system of factor analysis must yield the same values (apart from errors) for the factor loadings of individuals or tests, regardless of the combination in which they are presented." Others (Thurstone, 1947; Henrysson,

1957; and Quereshi, 1967) have assigned more than one meaning to invariance especially in those areas of psychology where factor analysis is regarded as the appropriate method of investigation. In such a context, invariance may have one of the following meanings:

- (a) consistency of factors, for a fixed set of variables, from one sample of individuals to another,
- (b) consistency of factors, for a fixed sample of individuals, from one set of variables to another,
- (c) consistency of factors when both samples and variables are methodically varied, and
- (d) consistency of factors, for a fixed sample of individuals and a fixed set of variables, from one method of analysis to another.

The present study is concerned with the first meaning ascribed to invariance.

NATURE OF THE PROBLEM

The problem of factorial invariance as stated by Hunka (1967) stems from the application of the factor analytic model to observed data. Since no unique mathematical solution exists for the model, the problem of determining when the same factors are present in two different studies is a major concern.

Problems involving estimation of factorial invariance may fall into one of four categories where each category is defined by (a) the nature of the sample and (b) the nature of the variables. The four conditions thus encountered and writers proposing solutions for each are presented in figure 1.

VARIABLES	
FIXED	VARIABLE
Wrigley-Neuhaus (1955)	Wrigley-Neuhaus (1955) Tucker (1958)
FIXED	
SAMPLE	A B
Burt (1948)	No adequate solution
Tucker (1951)	
Ahmavaara (1954) (1958)	
Wrigley-Neuhaus (1955)	
Kaiser (1960)	
Schönemann (1966)	
	C D
VARIABLE	

Figure 1

Types of Invariance and Proposed Solutions

SOLUTIONS TO FACTORIAL INVARIANCE

Although four types of invariance solutions can be distinguished, the only one of concern here is situation C - that of fixed tests and variable samples. In addition only those solutions using orthogonal transformations will be discussed.

If the same test is administered to two different samples, then configurational invariance may take place. Here the change in samples affects the size of the loadings, but in proportion to the changes in the variance of the tests administered to the different samples. Hence if one sample has less variance in some of the variables, this will result in a corresponding reduction in the size of the loadings. Although the numerical values are changed, the

configuration of the loadings should remain the same and the factors should be the same in all the analyses.

A number of attempts have been made to devise means for testing the type of invariance discussed above. In general, the problem could be approached along two lines. One could determine the relationship between the factors as vectors in the test space when the test variables are made to maximally overlap (Kaiser, 1960) or one could determine the degree of similarity of factors in terms of factor loadings. It is the latter approach that is employed in the present study. Furthermore, working with factor loadings and requesting orthogonal transformations opens another two avenues of approach. Starting with two sets of factor loadings, each of these could be rotated to a compromise position or one set could be defined as a target and the other set would be rotated to maximum congruence with it. Once again, the latter alternative is focal.

As was previously mentioned, various researchers have employed indices to assess the degree of factorial similarity. Perhaps one of the most common ways to assess stability of factors across samples is to calculate the "root mean square" suggested by Herman (1967). This index may be put in the form of

$$\sum_{j=1}^n \sqrt{(1^{a}_{jp} - 2^{a}_{jq})^2} / n$$

where 1^{a}_{jp} represents the factor loading of test j on factor p in study one. A similar interpretation can be given for 2^{a}_{jq} . This is a simple type of index and really does not indicate what comprises an adequate value for agreement. Furthermore factor one in study one could be

factor two in study two and the result would be an inappropriate pairing of factors.

Several investigators, Burt (1948), Tucker (1951), and Wrigley and Neuhaus (1955), have employed an index roughly resembling a correlation coefficient to compare loadings of tests on factors. Adopting Tucker's terminology such an index was called a coefficient of congruence and can be defined as

$$\frac{A'_1 A_2}{\sqrt{(A'_1 A_1) (A'_2 A_2)}}$$

The A 's represent matrices of factor loadings for studies one and two. The coefficient has a range for -1 to +1. Harman (1967) pointed out that this index is not a correlation since the loadings are not deviation scores and the summation is carried over the variables rather than the number of observations. In addition, tests of significance for the index are not available and neither Tucker nor Burt nor Wrigley and Neuhaus commit themselves as to state a minimal value for the coefficient.

To test the quality of fit, Kaiser employed a mean cosine index which can be represented as

$$\frac{1}{n} \text{ trace } (H_1^{-1} A_1' A_2' H_2^{-1})$$

where H_1 and H_2 represent the diagonal matrices composed of the square roots of the communalities for the common tests of the two studies and A_1 and A_2 represent the matrices of factor loadings.

Kaiser also does not specify a lower bound for the index.

Thus the aspect of defining indices which purport to measure

the degree of factorial similarity or stability appears to be meaningless if no guidelines are given to indicate what comprises a significant match. Using Schönemann's procedure it is the hope of the present study to establish such guidelines.

SCHÖNEMANN'S ORTHOGONAL PROCRUSTES SOLUTION

Schönemann (1966) showed how a source matrix A_1 of factor loadings could be transformed into a least squares approximation of a target matrix A_2 by an orthogonal matrix T . Green (1952) and Cliff (1966) have also provided solutions. Simply put, the problem is to find the matrix T given A_1 and A_2 . The situation can be represented as

$$A_1 T = A_2 + E \quad (1)$$

where E is the error matrix. The above equation can be rewritten as

$$E = A_1 T - A_2 \quad (2)$$

The Schönemann procedure yields a least-squares solution, that is,

$$\text{tr}(E'E) = \text{minimum} \quad (3)$$

under the constraint

$$T'T = TT' = I. \quad (4)$$

Schönemann went on to show that equation (3) can be expressed as

$$g_1 = \text{tr}(E'E) = \text{tr}(T'A_1'A_1T - 2T'A_1'A_2 + A_2'A_2) \quad (5)$$

The side condition $T'T = I$ is then expressed as

$$g_2 = \text{tr}(L(T'T - I)) \quad (6)$$

where L is a matrix of (unknown) Lagrange multipliers. Equations (5) and (6) are then expressed as

$$g = g_1 + g_2 \quad (7)$$

and g is partially differentiated with respect to the elements of the matrix T . The result is

$$\frac{\delta g}{\delta T} = (A_1' A_1 + A_1' A_1) T - 2A_1' A_2 + T(L + L') \quad (8)$$

If $A_1' A_1$ is represented by P , $A_1' A_2$ by S , and $(L + L')/2$ by Q , then one must solve

$$S = PT + TQ \quad (9)$$

Both P and Q are symmetric so that

$$Q = T'S - T'PT = Q' \quad (10)$$

Since $T'PT$ is symmetric if P is, then $T'S$ must be symmetric or

$$T'S = S'T \quad (11)$$

From (4) and (11) $S = TS'T$ and

$$SS' = TS'ST' \quad (12)$$

From $S = A_1' A_2$, two other matrices are formed such that

$$S'S = WDW' \quad (13)$$

and

$$SS' = VDV', \quad (14)$$

where D is the diagonal matrix of latent roots of both $S'S$ and SS' , W is the matrix of unit-length latent vectors of $S'S$, and V is the matrix of unit-length latent vectors of SS' .

Since $SS' = TS'ST'$, SS' can be substituted with WDW' and $S'S$ with VDV' yielding the equation

$$WDW' = TVDV'T' \quad (15)$$

Solving for W then yields

$$W = TV \quad (16)$$

or

$$T = W V' \quad (17)$$

Once T has been calculated, it is applied to matrix A_1 . The result A_1T represents the best approximation of the target matrix A_2 . The difference between A_1T and A_2 is computed and represents the error matrix, E . Since the transformation matrix T is unique it satisfies the criterion that $\text{tr}(E'E)$ is a minimum.

Obtaining $\text{tr}(E'E)$ and the fact that it is a minimum is a worthwhile criterion in comparing one set of factor loadings to another. Instead of assessing the values within the transformation matrix, an over all assessment of the approximation of the source matrix to the target matrix can be obtained. This over-all assessment is given by the single value of $\text{tr}(E'E)$.

However, the single value of $\text{tr}(E'E)$ offers very little knowledge as to the "quality of fit" of the source to the target matrix. A sampling distribution for the average $\text{tr}(E'E)$ would be helpful in assessing the "quality of fit" of the two data sets. It is the intent of the present study to determine a sampling distribution for the average $\text{tr}(E'E)$ values when the Orthogonal Procrustes solution is used for assessing factorial similarity.

SUMMARY

In this chapter several concepts of component analysis and factor matching were reviewed. Various indices to measure the degree of factorial similarity were presented. Finally, the procedure devised by Schönemann was described. His technique incorporated an orthogonal transformation matrix and matches a set of factor loadings called a source to another set of loadings called the target matrix.

An outline of the procedure will be presented in Chapter III.

CHAPTER III

PROCEDURE

Twenty sets of one thousand random variables were generated. Each set was approximately normally distributed with a mean of fifty and a standard deviation of five. A correlation matrix (R) of order twenty by twenty was calculated and factored, from which twenty uncorrelated sets of one-thousand component scores each were produced. Each set had a mean of zero and a standard deviation of one. This matrix of component scores (F) was of order one-thousand by twenty and served as the source for the samples used in the study. Various component structure matrices (A) of order n variables by r components were selected from the literature for use as the population matrices. By multiplying $Z = A F'$ we would have in Z , the scores of 1,000 people on n variables such that if a component analysis were performed on the correlation matrix derived from the scores of Z , an approximation of the structure matrix A would be returned. For the remaining part of this section an A matrix of order five variables by three components will be used as an example to add clarity to the procedure.

Two samples of fifty component scores ($\hat{F}$) each were drawn with replacement from the first three columns of the population matrix (F). These two samples of order fifty observations by three components were designated as $\hat{F}_1$ and $\hat{F}_2$. For each sample of fifty component scores, a product matrix was produced by forming $\hat{Z} = A \hat{F}'$. Thus for the two samples drawn each $\hat{F}$ was premultiplied by the A matrix of order five variables by three components resulting in

$$\hat{Z}_1 = A \hat{F}'_1$$

and

$$\hat{Z}_2 = A \hat{F}'_2$$

where both $\hat{Z}_1$ and $\hat{Z}_2$ were of the order fifty observations by five variables. The procedure thus far is based on the idea that two samples of size fifty drawn from $Z = A F'$ (that is, the population) is the same as drawing two samples of $\hat{F}$ and forming $\hat{Z} = A \hat{F}'$.

For each $\hat{Z}$ thus formed, correlations were computed among the five variables to produce a correlation matrix ($\hat{R}$) of order five variables by five variables. A component structure matrix ($\hat{A}$) was calculated for each $\hat{Z}$ matrix. Thus for $\hat{Z}_1$ a corresponding $\hat{A}_1$ was calculated and likewise an $\hat{A}_2$ for $\hat{Z}_2$. Each A was of the order five variables by three components, and the two $\hat{A}$'s were produced by two samples coming from the same population. Employing Schönemann's procedure the two component structure matrices $\hat{A}_1$ and $\hat{A}_2$ were matched. Additional factor score samples of order fifty by three were drawn pairwise and the A matrix was used to produce additional $\hat{Z}$ matrices which were factored and compared until one thousand matches had been performed.

For each match the largest absolute value of the error matrix and average $\text{tr}(E'E)$ were computed. Average $\text{tr}(E'E)$ is defined by

$$\frac{1}{nr} \sum_{i=1}^n \sum_{j=1}^r e_{ij}^2$$

where

e = elements of the error matrix

n = number of variables

r = number of components.

Hence for every A matrix used, one-thousand "largest absolute value" and one-thousand average $\text{tr}(E'E)$ values of the error matrix were obtained. These one-thousand values were an empirical sampling distribution given the true null hypothesis that $\hat{A}_1$ and $\hat{A}_2$ are both estimates of a common A matrix. Table 1 shows the various A matrices which were used in the study. References for the A matrices may be found in Appendix A.

For several of the A matrices, the procedure was repeated using sample sizes of 100 and 150. In such cases only 500 matches were performed rather than the 1000. Matrices subjected to the increased sample sizes are presented in Table 2.

Frequency distributions of the "largest absolute value" and the average $\text{tr}(E'E)$ values for the different A matrices were obtained. For each distribution, the 25, 50, 75, 90, 95, and 99 percentile points were calculated. In addition the maximum value of the "largest absolute value" and the average $\text{tr}(E'E)$ values were obtained. Cumulative frequency polygons for the average $\text{tr}(E'E)$ for the 5×3 , 10×4 , 16×3 , and 20×6 matrices were plotted. Frequency polygons were also plotted for several of the matrices subjected to the increased sample sizes. For such cases, plots appear for Harman's 5×3 , Ohnmacht's 16×3 , and Evanetchko's 20×6 factor structure matrices.

TABLE 2
MATRICES SUBJECTED TO SAMPLE SIZES
OF 100 AND 150

Order	Reference
5 x 3	Harman
	Morrison
8 x 2	Harman
	Harman
10 x 4	Mosychuk
	Noble
16 x 3	Hallworth
	Ohnmacht
20 x 6	Hallworth
	Evanechko

SUMMARY

An outline of the procedure incorporated by the study was presented in the present chapter. The study adopts for its basis a randomly generated factor score matrix of order one thousand observations by twenty variables and twenty-two different A matrices ranging in order from five by three to twenty by six.

Results of the study will be presented in the following chapter.

CHAPTER IV

RESULTS

Table 3 shows the percentile points and maximum value for average $\text{tr}(\mathbf{E}'\mathbf{E})$ of the various A matrices used in the study. Identical percentile point values calculated for the "largest absolute value" in the error matrix are presented in Table 4. Similar information is presented in Tables 5 and 6 for the A matrices whose sample sizes were increased to 100 and 150. Percentile values for a sample size of 50 are also included in these tables for comparative purposes.

Figures 2, 3, 4, and 5 are the cumulative frequency polygons for average $\text{tr}(\mathbf{E}'\mathbf{E})$ for the 5 by 30, 10 by 4, 16 by 3, and 20 by 6 A matrices respectively. Frequency polygons for the same A matrices appear in Figures 6, 7, 8, and 9. Frequency polygons for data subjected to sample sizes of 50, 100, and 150 appear in Figures 10, 11, and 12 respectively.

DISCUSSION OF THE RESULTS

If the Orthogonal Procrustes solution is used as the factor matching technique, then Tables 3, 4, 5, and 6 should provide some guidelines as to the "quality of fit" of two data sets. Of practical importance are the values obtained for the 95 and 99 percentiles and maximum values of average $\text{tr}(\mathbf{E}'\mathbf{E})$ and the "largest absolute value" of the error matrix. At the 95 percentile, values for average $\text{tr}(\mathbf{E}'\mathbf{E})$ range from 0.0074 to 0.0145. At the 99 percentile the values range from 0.0112 to 0.0232 while the maximum average $\text{tr}(\mathbf{E}'\mathbf{E})$ values

TABLE 4

SELECTED PERCENTILE POINTS AND MAXIMUM VALUE OF THE
CUMULATIVE FREQUENCY FOR "LARGEST ABSOLUTE
VALUE" IN THE ERROR MATRIX

(Leading decimal points omitted)

A Matrix Source	Order	Percentile						Maximum Value
		25	50	75	90	95	99	
Harman	5x3	0842	1176	1580	1983	2364	2812	3600
Hunka	5x3	1040	1426	1837	2246	2538	3193	4200
Morrison	5x3	0691	0976	1276	1597	1824	2456	3000
Morrison	5x3	1135	1528	1950	2375	2760	3300	4200
Morrison	6x3	1175	1621	2124	2598	2912	3471	4050
Morrison	6x3	0891	1278	1737	2187	2543	3337	4500
Harman	8x2	0736	1215	1816	2534	2953	3825	5250
Harman	8x2	0696	1145	1660	2154	2561	3100	4050
Mosychuk	10x4	1625	2030	2460	2954	3276	3870	4350
Noble	10x4	1658	2088	2620	3176	3529	4350	5100
Kraus & Walker	11x6	1919	2222	2593	2977	3246	3600	4650
Hogg	12x4	1446	1849	2283	2830	3142	3900	4800
Eyre	12x4	1517	1925	2379	2875	3270	3925	4950
Lovell & Gorton	12x6	1812	2222	2683	3207	3573	4387	5700
Wiggins & Lovell	13x3	1426	1848	2335	2875	3246	3950	5250
Morrison	13x4	1403	1774	2248	2680	3028	3690	4650
Hallworth	16x3	1371	1765	2209	2724	2982	3525	4800
Ohnmacht	16x3	1508	2120	2854	3620	4217	5137	6750
Taylor & Maguire	18x3	1258	1668	2234	2902	3339	4400	5400
Walberg	18x5	2088	2511	3019	3515	3867	4500	5850
Hallworth	20x6	2069	2486	2909	3435	3700	4512	4950
Evanechko	20x6	2161	2532	2997	3219	3702	4320	5550

TABLE 5

SELECTED PERCENTILE POINTS AND MAXIMUM VALUE
FOR AVERAGE $\text{tr}(\mathbf{E}'\mathbf{E})$ FOR VARIOUS SAMPLE SIZES
(Leading decimal points omitted)

Order and Reference											
		5 x 3		8 x 2		10 x 4		16 x 3		20 x 6	
Percentile	Sample Size	Harman	Morrison	Harman (E)	Harman (P)	Mosychuk	Noble	Hallworth	Ohnmacht	Hallworth	Evanechko
25	50	0015	0027	0008	0009	0041	0031	0030	0016	0045	0058
	100	0005	0014	0005	0006	0019	0013	0015	0007	0021	0027
	150	0004	0008	0005	0005	0012	0073	0084	0005	0015	0019
50	50	0029	0046	0018	0023	0061	0045	0051	0028	0060	0073
	100	0012	0024	0010	0012	0028	0022	0025	0014	0028	0037
	150	0009	0017	0009	0010	0020	0015	0017	0011	0021	0024
75	50	0054	0075	0037	0049	0084	0065	0081	0046	0076	0093
	100	0019	0037	0017	0026	0041	0030	0040	0025	0039	0046
	150	0013	0027	0014	0014	0028	0024	0028	0017	0027	0030
90	50	0089	0104	0063	0086	0112	0087	0121	0069	0095	0114
	100	0030	0055	0033	0045	0055	0043	0059	0038	0049	0057
	150	0019	0040	0024	0027	0037	0029	0041	0026	0031	0040
95	50	0119	0132	0087	0111	0133	0102	0145	0085	0110	0127
	100	0039	0068	0042	0063	0064	0052	0072	0048	0056	0062
	150	0026	0045	0031	0036	0042	0036	0050	0029	0039	0043
99	50	0189	0202	0131	0178	0175	0133	0210	0118	0152	0149
	100	0060	0097	0063	0095	0086	0067	0107	0067	0072	0074
	150	0039	0066	0054	0063	0053	0045	0072	0041	0045	0049
Maximum	50	0330	0345	0210	0420	0225	0210	0300	0225	0210	0210
	100	0075	0135	0110	0165	0105	0075	0120	0090	0105	0090
	150	0075	0105	0090	0090	0075	0090	0090	0045	0075	0060

TABLE 6
SELECTED PERCENTILE POINTS AND MAXIMUM VALUE OF
"LARGEST ABSOLUTE VALUE" IN THE ERROR MATRIX
FOR VARIOUS SAMPLE SIZES
(Leading decimal points omitted)

Order and Reference											
		5 x 3		8 x 2		10 x 4		16 x 3		20 x 6	
Percentile	Sample Size	Harman	Morrison	Harman (E)	Harman (P)	Mosychuk	Noble	Hallworth	Ohnmacht	Hallworth	Evanechko
25	50	0842	1135	0696	0736	1625	1658	1371	1508	2068	2161
	100	0483	0809	0460	0515	1118	1095	0929	1056	1470	1536
	150	0380	0673	0439	0405	0910	0897	0772	0897	1214	1159
50	50	1176	1528	1145	1215	2030	2088	1765	2120	2486	2532
	100	0668	1050	0754	0856	1386	1369	1236	1500	1723	1782
	150	0544	0885	0652	0661	1163	1165	1022	2182	1422	1448
75	50	1580	1950	1660	1816	2460	2620	2209	2854	2909	2997
	100	0893	1374	1113	1286	1711	1732	1571	1991	2060	2097
	150	0723	1133	0941	0963	1384	1459	1267	1800	1656	1662
90	50	1983	2375	2154	2534	2954	3176	2724	3620	3435	3419
	100	1150	1721	1517	1800	2057	2118	1934	2686	2377	2421
	150	0879	1442	1279	1344	1601	1795	1546	2166	1905	1884
95	50	2365	2765	2461	2953	3276	3529	2982	4217	3700	3702
	100	1312	1940	1815	2088	2378	2400	2170	3125	2675	2655
	150	0986	1586	1562	1775	1762	1967	1746	2342	2111	2060
99	50	2812	3300	3100	3825	3870	4350	3525	5137	4512	4320
	100	1620	2300	2280	2650	2925	3037	2490	3825	3125	3100
	150	1275	1850	1970	2190	2062	2550	2130	2850	2475	2400
Maxi-mum	50	3600	4200	4050	5250	4350	5100	4800	6750	4950	5550
	100	1950	2700	2700	4050	3300	3600	2850	4500	3900	3450
	150	1650	2700	3150	2550	2550	3000	2400	3450	3150	2850

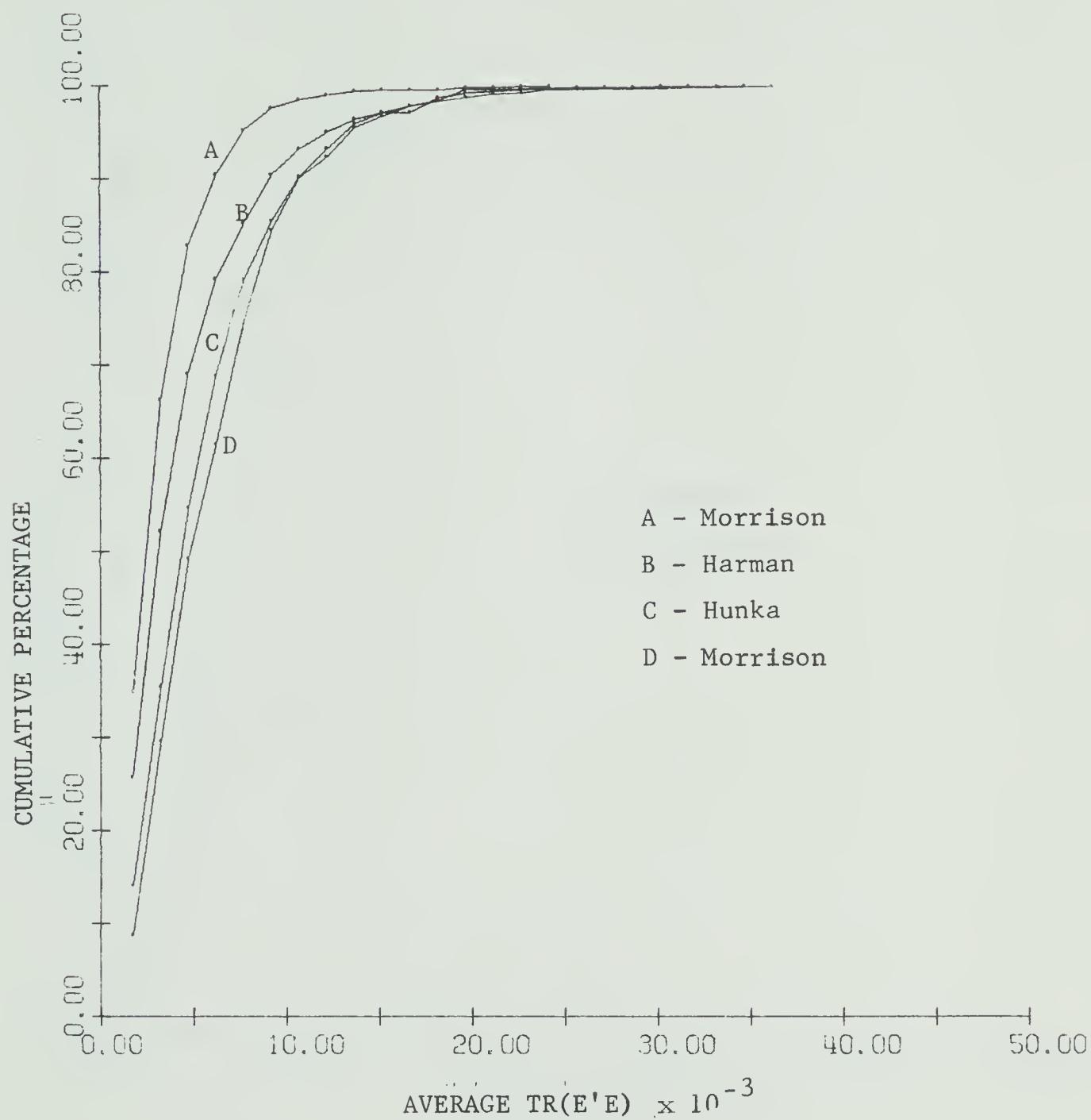


FIGURE 2 CUMULATIVE POLYGONS FOR AVERAGE TRACE E'E
VALUES USING VARIOUS 5 BY 3 A MATRICES

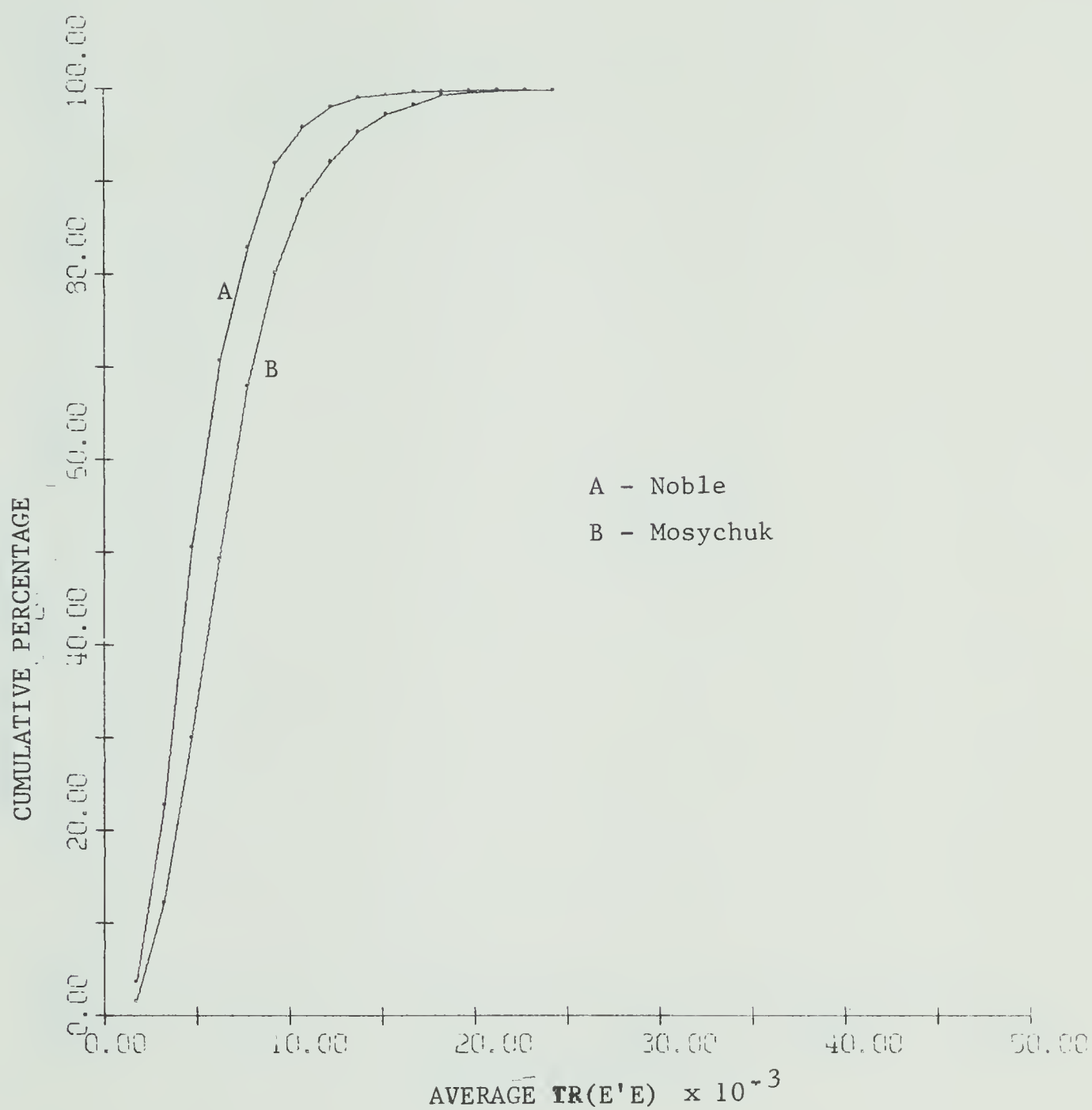


FIGURE 3 CUMULATIVE FREQUENCY POLYGONS FOR AVERAGE TRACE $\text{E}'\text{E}$
VALUES USING VARIOUS 10 BY 4 A MATRICES

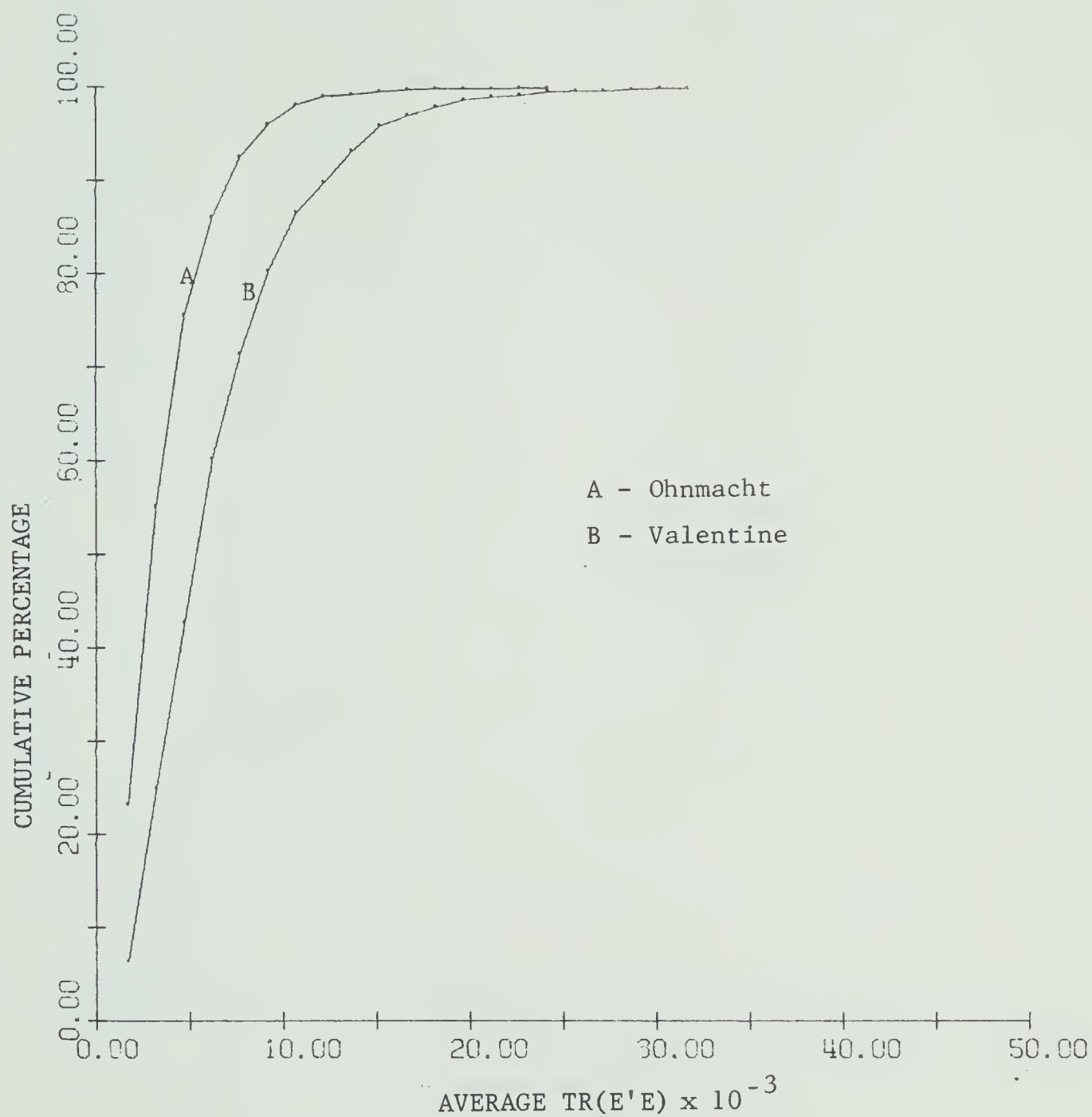


FIGURE 4 CUMULATIVE FREQUENCY POLYGONS FOR AVERAGE TRACE E'E
VALUES USING VARIOUS 16 BY 3 A MATRICES

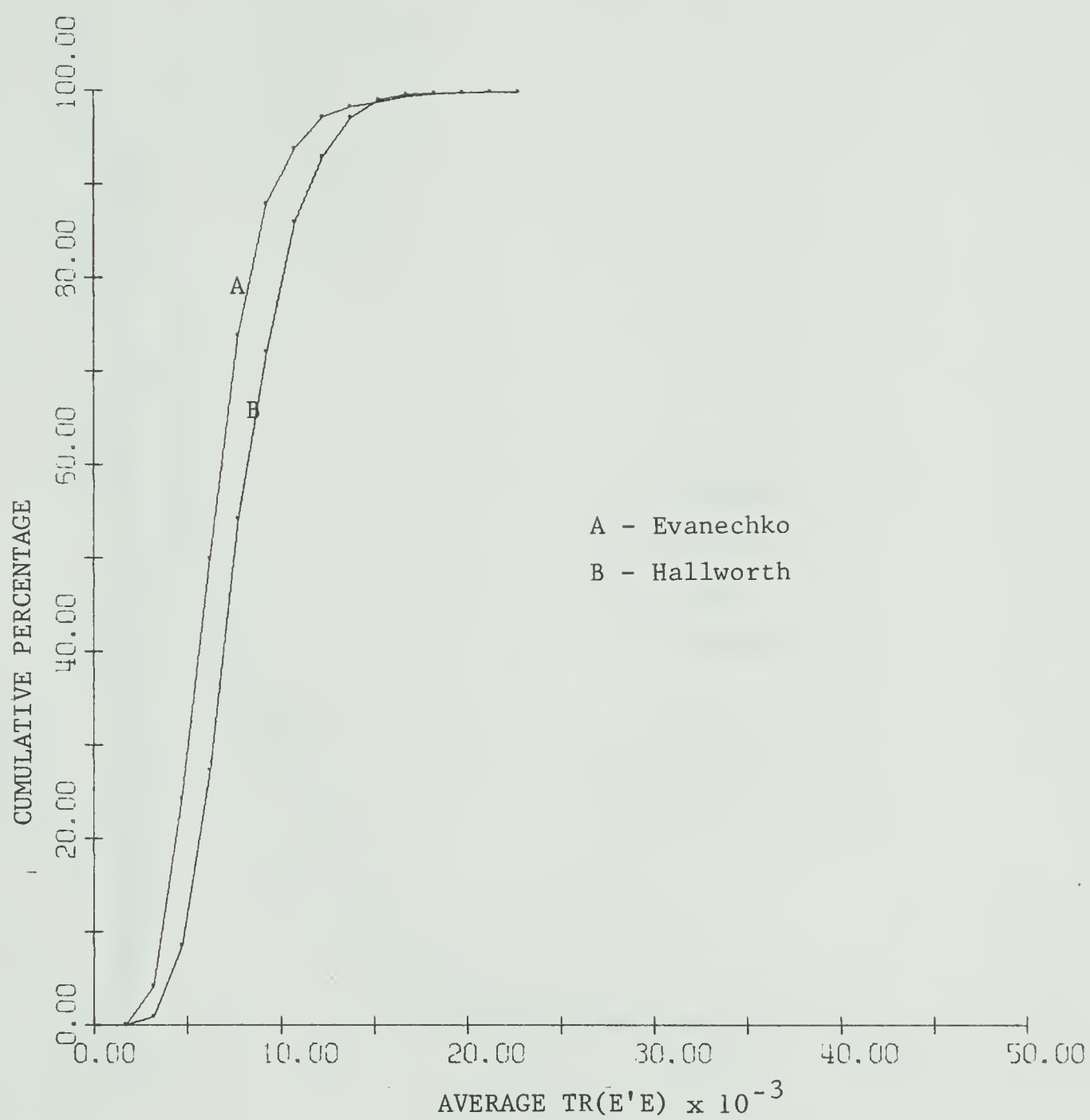


FIGURE 5 CUMULATIVE FREQUENCY POLYGONS FOR AVERAGE TRACE $E'E$
VALUES USING VARIOUS 20 BY 6 A MATRICES

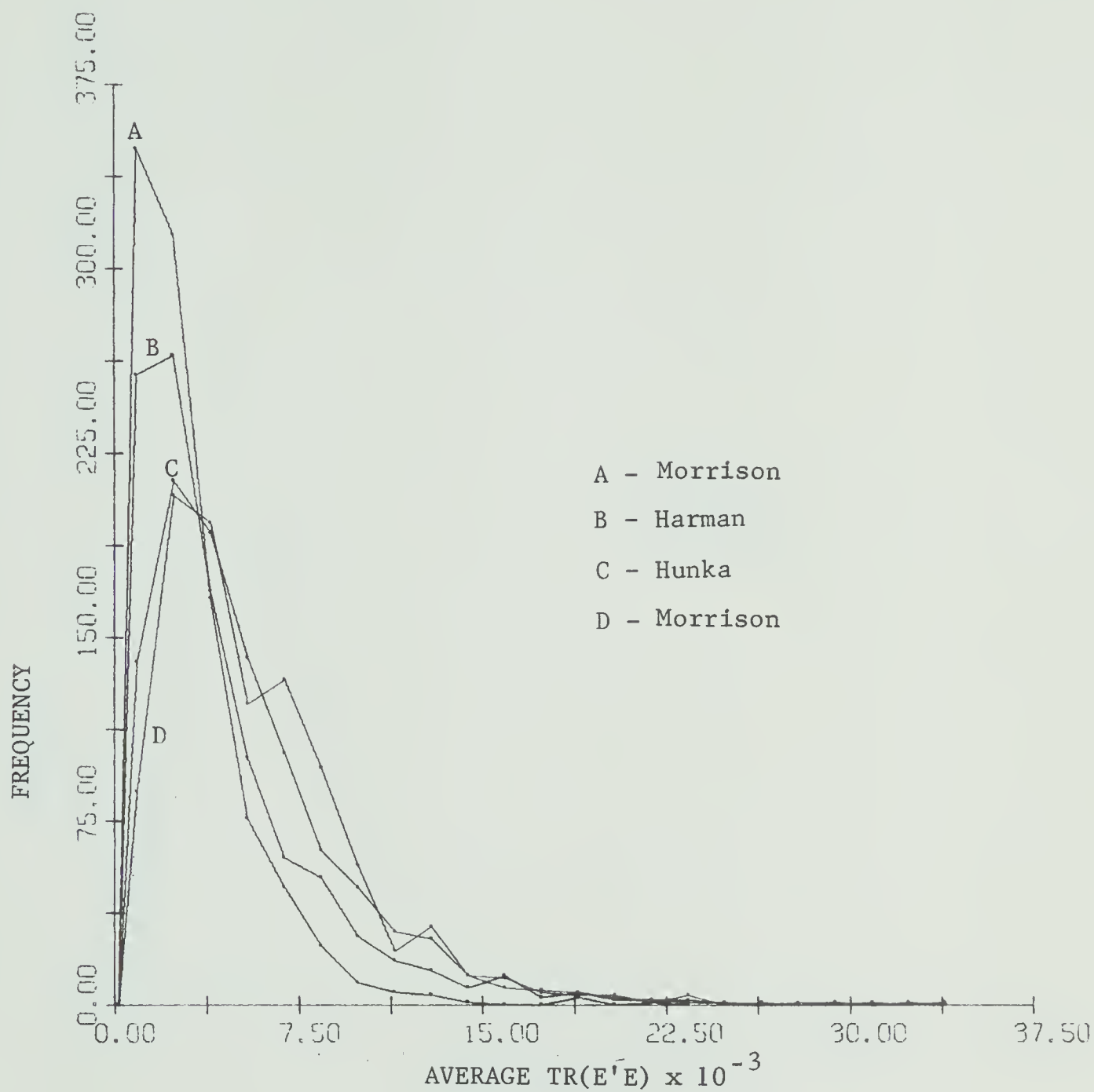


FIGURE 6 FREQUENCY POLYGONS FOR AVERAGE TRACE $E'E$
VALUES USING VARIOUS 5 BY 3 A MATRICES

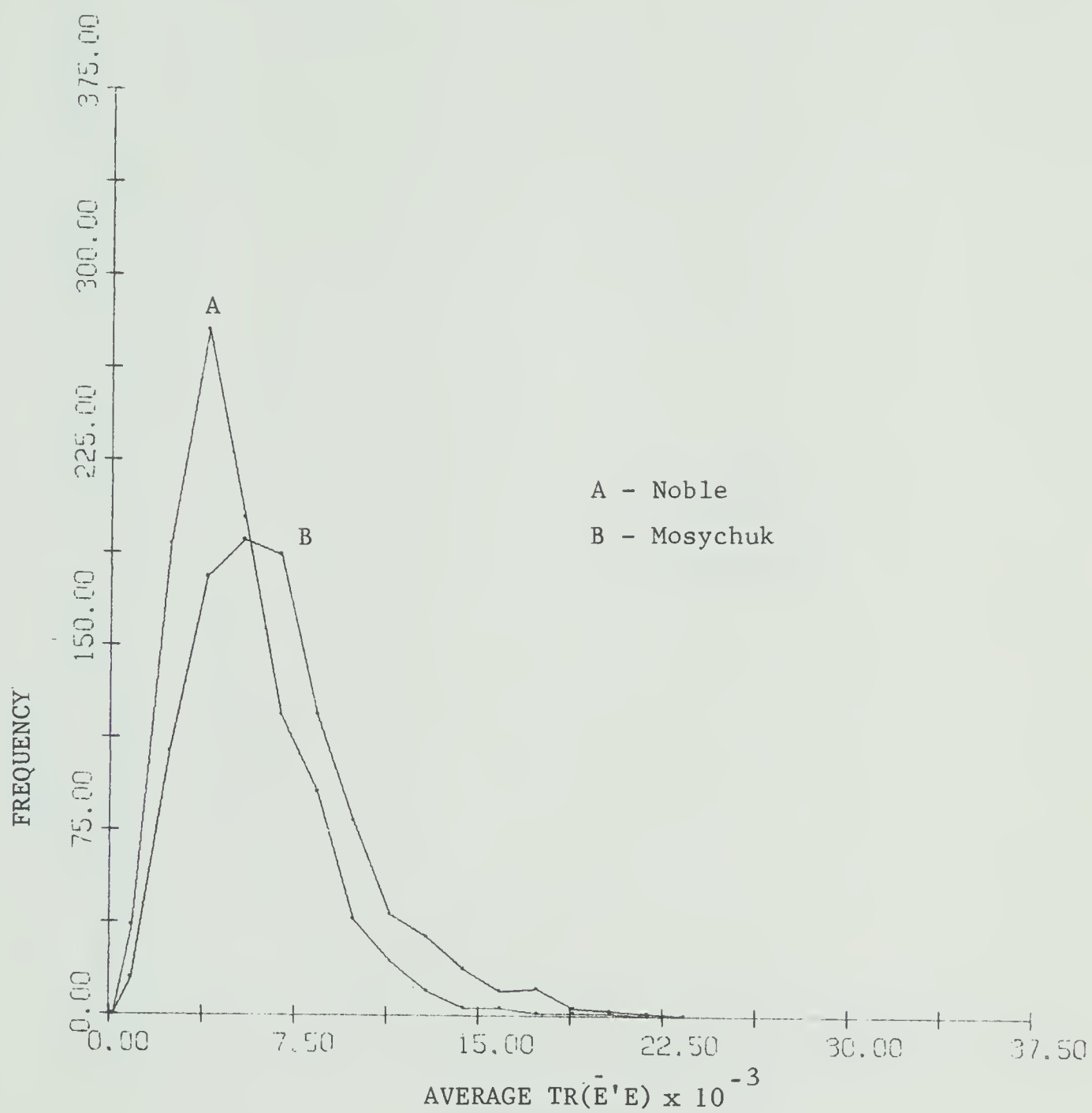


FIGURE 7 FREQUENCY POLYGONS FOR AVERAGE TRACE $E'E$
VALUES USING VARIOUS 10 BY 4 A MATRICES

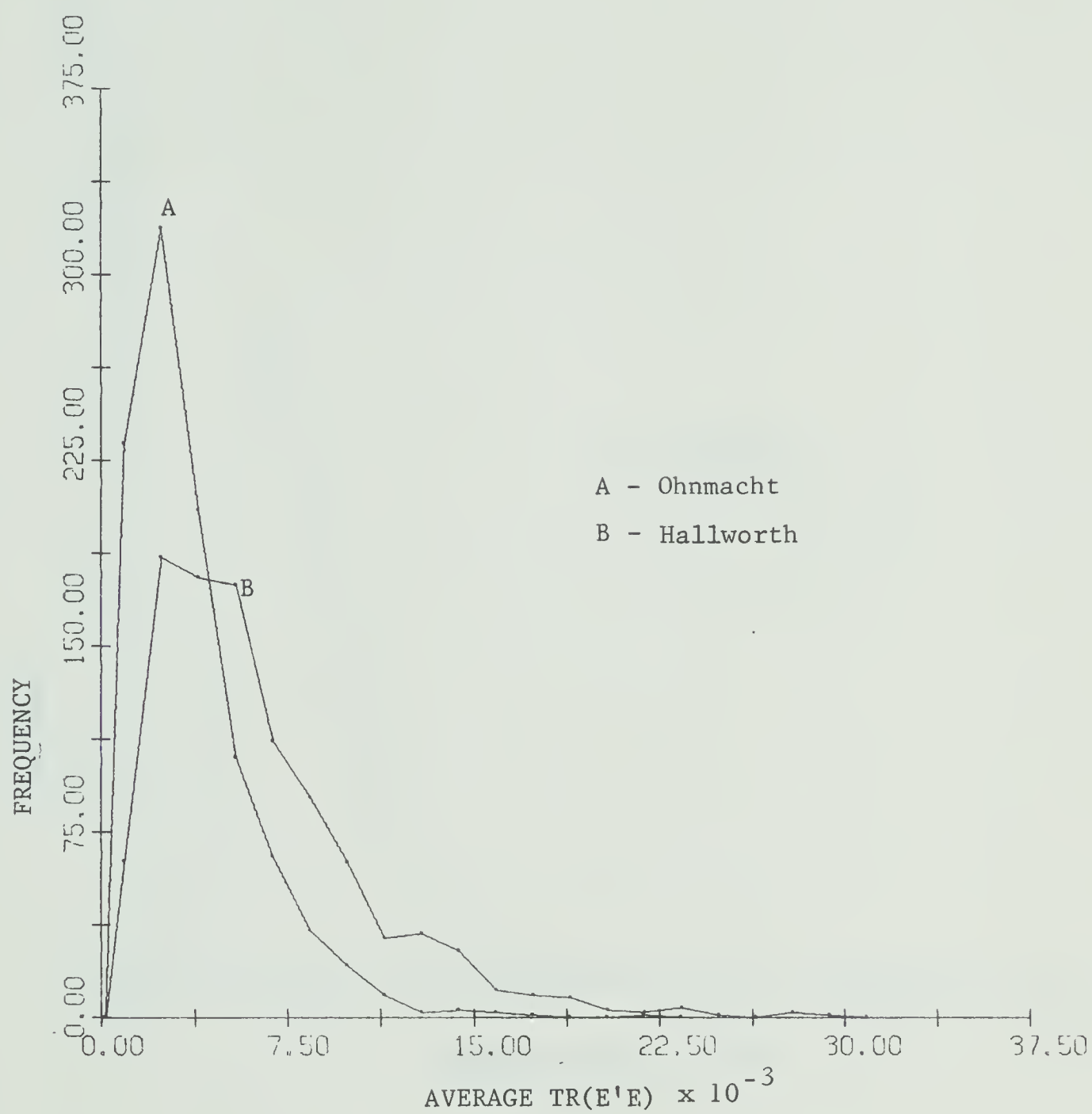


FIGURE 8 FREQUENCY POLYGONS FOR AVERAGE TRACE $E'E$
VALUES USING VARIOUS 16 BY 3 A MATRICES

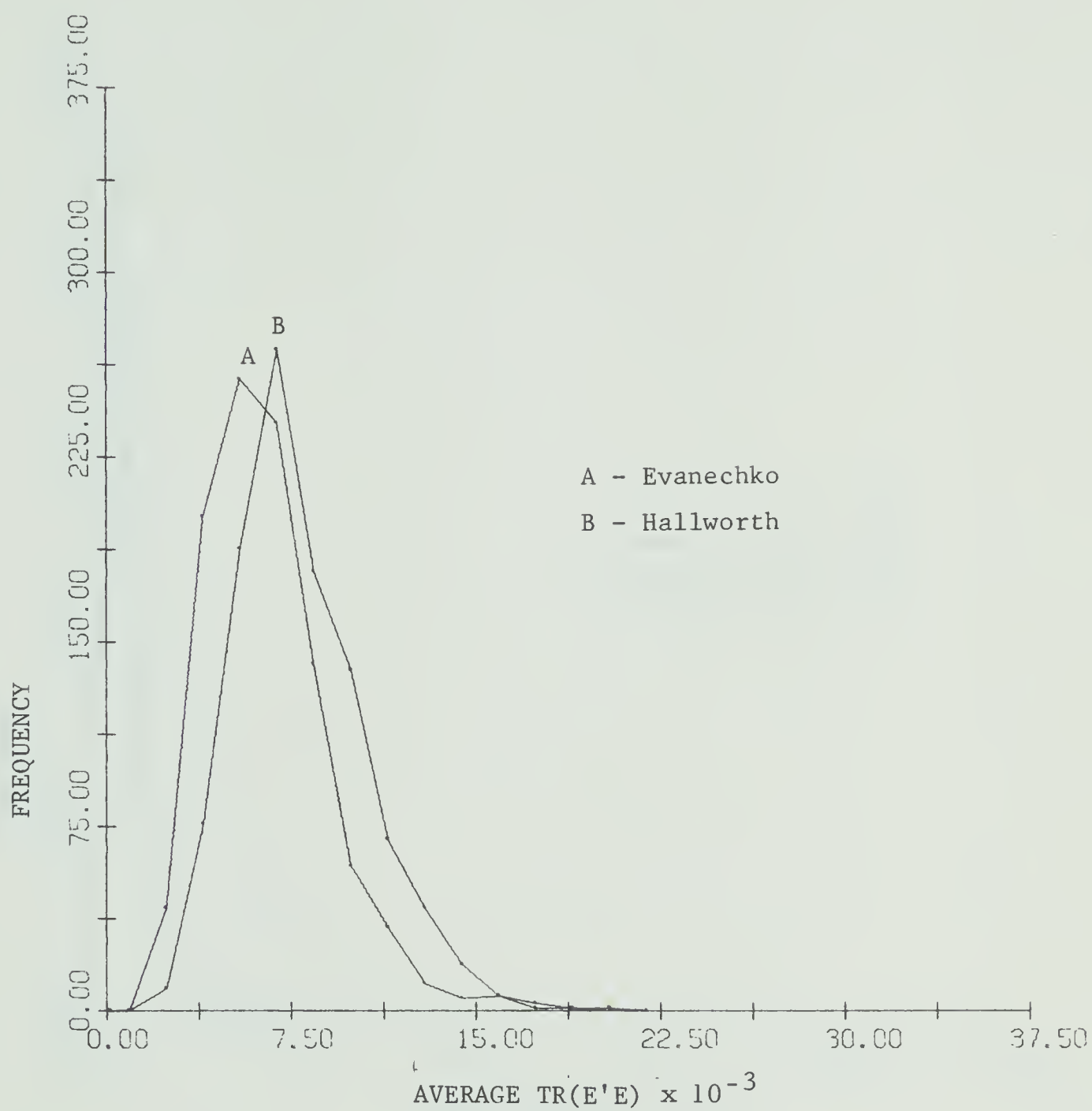


FIGURE 9 FREQUENCY POLYGONS FOR AVERAGE TRACE $E'E$
VALUES USING VARIOUS 20 BY 6 A MATRICES

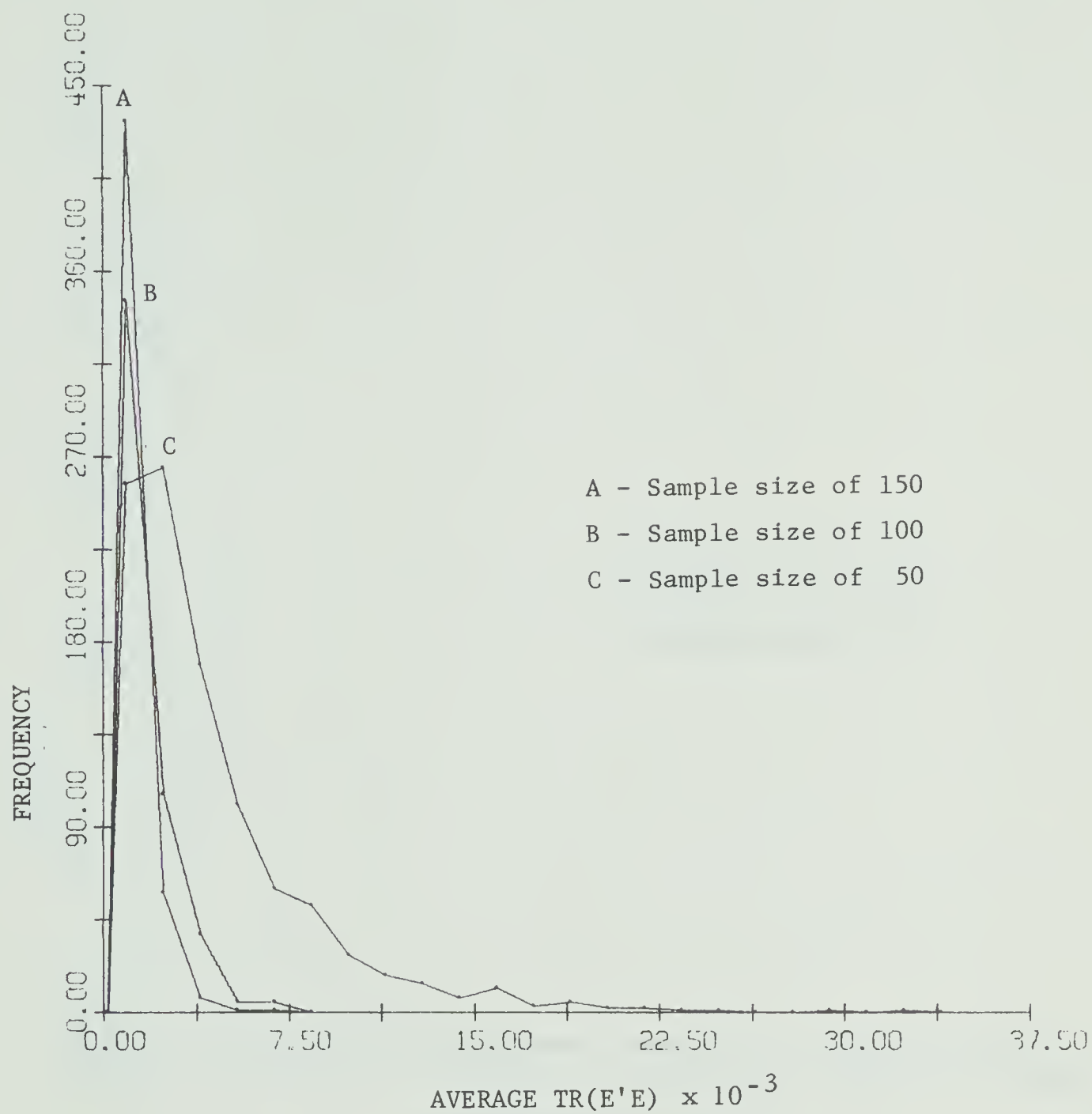


FIGURE 10 FREQUENCY POLYGONS FOR AVERAGE TR E'E VALUES OF HARMAN 5 BY 3
 MATRIX USING SAMPLE SIZES OF 50 100 AND 150

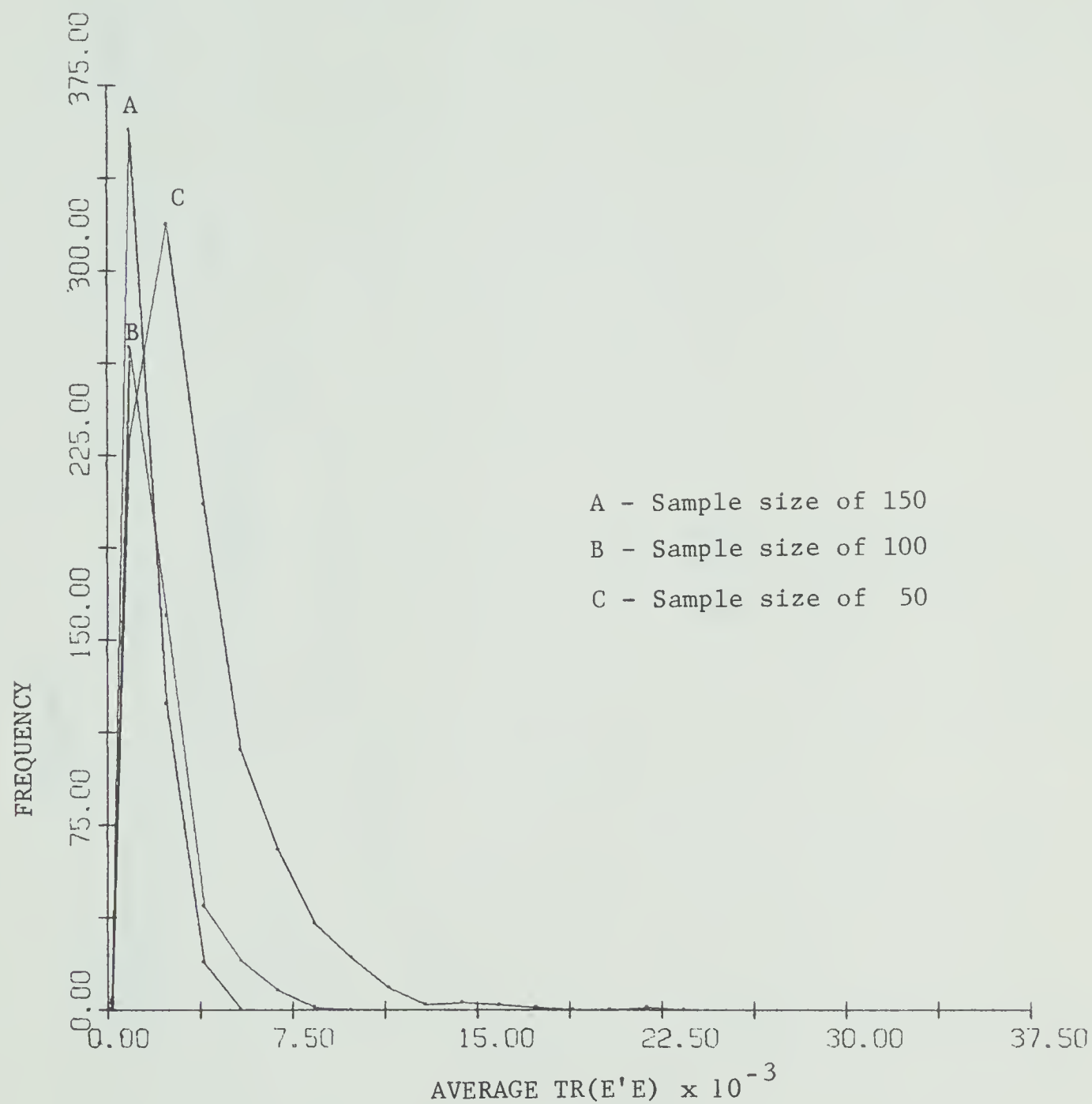


FIGURE 11 FREQUENCY POLYGONS FOR AVERAGE TR E'E VALUES OF OHNMACHT 16 BY 3
MATRIX USING SAMPLE SIZES OF 50 100 AND 150

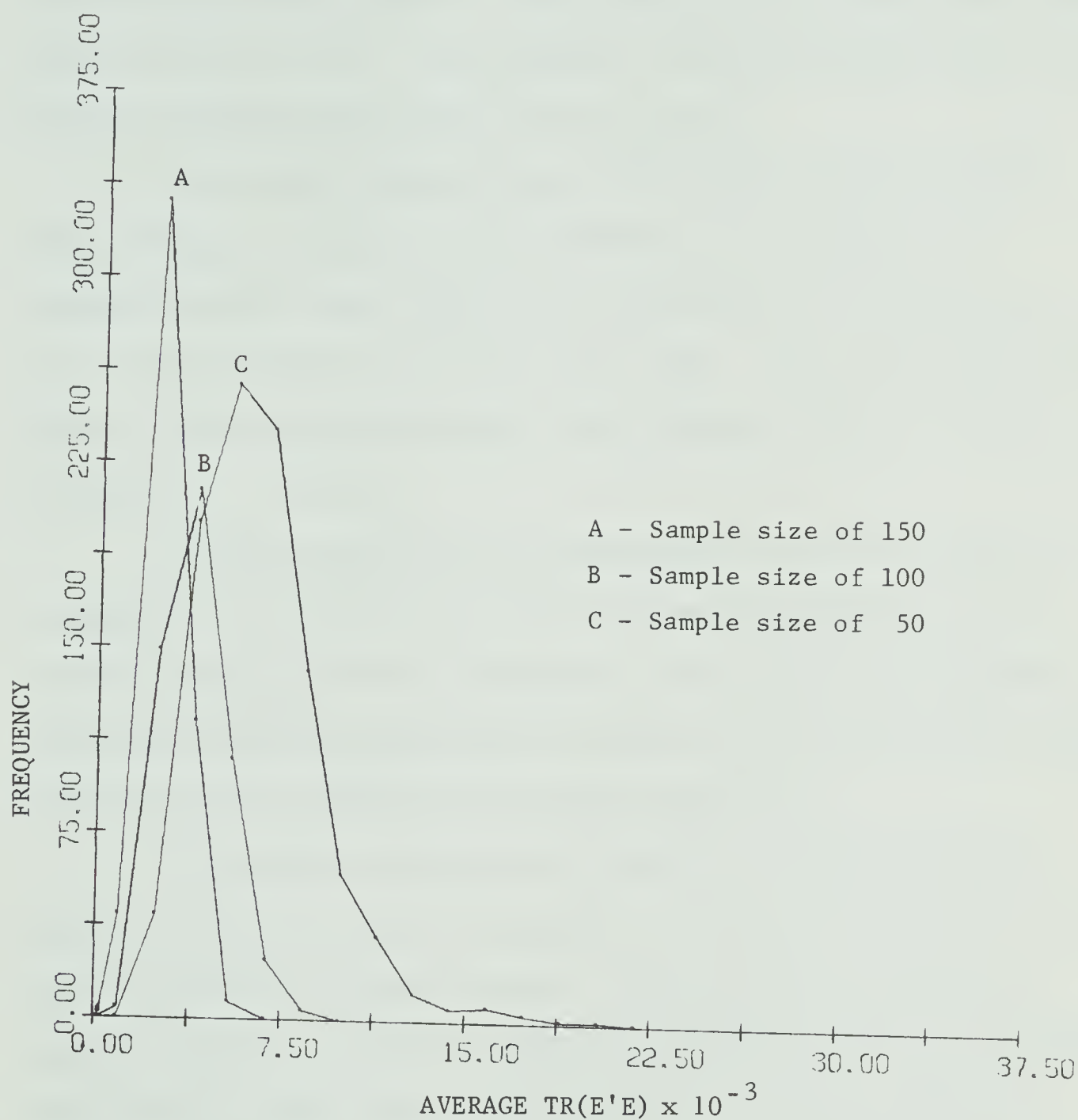


FIGURE 12 FREQUENCY POLYGONS FOR AVERAGE $TR E'E$ VALUES OF
 EVANECHKO 20 BY 6 MATRIX USING SAMPLE SIZES
 OF 50 100 AND 150

over the twenty-two A matrices range from 0.0135 to 0.0420. For the "largest absolute value" in the error matrix, the 95 percentile values range from 0.1824 to 0.14217 while for the 99 percentile range from 0.25456 to 0.5137. The maximum "largest absolute value" for the error matrix range from 0.3000 to 0.6750.

Observation of the average $\text{tr}(E'E)$ and "largest absolute value" values revealed no trend (stability of average $\text{tr}(E'E)$ or "largest absolute value") either over the twenty-two A matrices or over matrices which were of the same order. The lack of trend was further enhanced by observation of the frequency polygons depicted in Figures 6, 7, 8 and 9.

Although graphs were not presented for all of the matrices, a rough plot of all twenty-two matrices revealed that all the curves would be positively skewed. Since the criterion for the Orthogonal Procrustes solution required that average $\text{tr}(E'E)$ be a minimum then positively skewed distributions would be expected.

A plausible explanation for variations in the distribution would be the use of different A matrices. Since the A matrices were chosen without any specific criteria, aside from an unrotated or a varimax rotation, variations would be dependent upon the correlation matrix which produced the corresponding A matrices. Hence one would have the intuitive feeling that Harman's correlation matrix of five socio-economic variables would produce a different factor structure matrix than Hunka's correlation matrix of five textbook variables. Each correlation matrix is characterized by two characteristics--composition of the sample and composition of the

variables. Composition of the sample could be further partialled into two facets: number and type of people. An increase in the sample size and whether the people were homogeneous or heterogeneous with respect to some entity would affect the correlation matrix and consequently the component structure. Similar changes in the composition of the variables would also affect the correlation matrix. Therefore, if the parameters of the correlation matrix are changed, characteristics of the component structure matrix will also be changed.

Tables 5 and 6 presented the changes in the magnitude of average $\text{tr}(\mathbf{E}'\mathbf{E})$ and the "largest absolute value" for ten of the twenty-two $\mathbf{A}$ matrices when sample size was increased from 50 to 100 and 150. At the 95 percentile and sample sizes of 50, 100, and 150, average $\text{tr}(\mathbf{E}'\mathbf{E})$ values ranged from 0.0085 to 0.0145, 0.0048 to 0.0072, and 0.0026 to 0.0050 respectively. Values at the 99 percentile and identical samples sizes ranged from 0.0131 to 0.0210, 0.0060 to 0.0107, and 0.0039 to 0.0072 and maximum average $\text{tr}(\mathbf{E}'\mathbf{E})$ values ranged from 0.0210 to 0.0345, 0.0075 to 0.0165, and 0.0045 to 0.0105. The range for the "largest absolute value" at the 95 percentile and sample sizes of 50, 100, and 150 were 0.2365 to 0.4217, 0.1312 to 0.3125, and 0.0986 to 0.2342 for the respective sample sizes. Values ranged from 0.2812 to 0.5137, 0.1620 to 0.3825, and 0.1275 to 0.2850 at the 99 percentile and from 0.3600 to 0.6750, 0.1950 to 0.4500, and 0.1650 to 0.3450 for the maximum value. Hence values for average $\text{tr}(\mathbf{E}'\mathbf{E})$ and "largest absolute value" decreased in magnitude with increases in sample size. This conclusion strengthened previous statements that increases in the

sample size would affect the correlation matrix. An increase in the sample size yields a better estimate of whatever population parameter is being measured. Factoring these correlation matrices produces less error and thereby the better the goodness of fit between pairs of factor structures. Figures 10, 11, and 12 show the trend of a decrease in the magnitude of average $\text{tr}(\mathbf{E}'\mathbf{E})$ with an increase in the sample size.

SUMMARY

The results of the study were reported in the present chapter. Tables were presented to indicate various percentile points and maximum values for average $\text{tr}(\mathbf{E}'\mathbf{E})$ and "largest absolute value" for twenty-two different $\mathbf{A}$ matrices. Percentile points were also reported for ten of the $\mathbf{A}$ matrices which were subjected to increased sample sizes. In the Discussion of the Results, an attempt was made to summarize points of practical importance. In addition to the numerical results, several graphs were included to give an indication of the distribution and shape of the average $\text{tr}(\mathbf{E}'\mathbf{E})$ values for several of the $\mathbf{A}$ matrices.

Summary, conclusions, and recommendations of the study will be presented in the final chapter.

CHAPTER V

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

SUMMARY

In the present study an attempt was made at determining an empirical sampling distribution from one type of factor match technique--the Orthogonal Procrustes solution. One-thousand matches were performed for each of twenty-two different A matrices. Frequency distributions of average $\text{tr}(\mathbf{E}'\mathbf{E})$ and "largest absolute value" were computed for each set of one thousand matches. The 25, 50, 75, 90, 95, and 99 percentile points were calculated for each distribution. In addition to these values the maximum value for distributions was calculated. The same procedure was repeated for several A matrices which were subjected to sample size increases. Percentile points and maximum value for average $\text{tr}(\mathbf{E}'\mathbf{E})$ and "largest absolute value" were reported. Several graphs of average $\text{tr}(\mathbf{E}'\mathbf{E})$ for some of the A matrices were also plotted.

CONCLUSIONS

There is a certain amount of constraint imposed on generalizations from the present study. Because a particular set of methodological procedures were implemented and a certain set of parameters held constant, generalization of results from this study to empirical studies employing other methods should be undertaken with caution. Different methods of factoring and factor matching as well as different number of variables and components could produce different results from those reported here.

If factor structure matrices are a result of a principal component analysis and are either unrotated or rotated to the varimax criterion and later subjected to an Orthogonal Procrustes solution for factor matching, results of this study could be used as guidelines in evaluating the quality of fit. Although no probabilities can be associated with the various percentile levels reported, it would appear that an average $\text{tr}(\mathbf{E}'\mathbf{E})$ value greater than 0.0232 (99 percentile) would indicate that the pair of factor structures are different. In addition, average $\text{tr}(\mathbf{E}'\mathbf{E})$ values at the 95 percentile (0.0145), 90 percentile (0.0121), and the 75 percentile (0.0093) may be used for assessing the significance of factorial similarity. The above mentioned average $\text{tr}(\mathbf{E}'\mathbf{E})$ values for the various percentiles result from a sample size of fifty. Table 5 reveals that sample size increases would decrease the magnitude of $\text{tr}(\mathbf{E}'\mathbf{E})/\text{nr}$.

Some researchers might want to use the "largest absolute value" as an index of similarity between two factorial structures. A "largest absolute value" greater than 0.5137 (99 percentile) would show that the two structures are different. "Largest absolute values" at the 95, 90, and 75 percentile levels may be used as guidelines in determining whether the structures are similar. Researchers using the "largest absolute value" as a measure of factorial similarity should be aware of the effect of increased sample sizes upon the magnitude of the "largest absolute value."

Even though it is possible to have factorial similarity and an average $\text{tr}(\mathbf{E}'\mathbf{E})$ greater than 0.023 or a "largest absolute value" greater than 0.5137, heuristic evidence from this study would

suggest that the structures are different. If the results of the present study are used in the interpretation of say the Taylor and Maguire data and we adopt the 90 percentile as our level of confidence, then the null hypothesis of 'no major differences exist between the groups' perceptions of science objectives' is rejected. The conclusion is based on the fact that the observed average $\text{tr}(E'E)$ value of 0.015 is greater than the critical value of 0.012 at the 90 percentile.

RECOMMENDATIONS

Whereas the results of the present study have been encouraging, several modifications could be implemented for an extension of research in this area. Sample sizes larger than 150 could be used in the analysis.

Since the results of the study are specific to the A matrices used, A matrices could be produced randomly. Kaiser and Dickman (1962) and Cliff and Pennell (1967) have outlined methods for generating sample correlation matrices based on a population correlation matrix. Results thus obtained could be compared with the findings of the present study. Since the effect of sample size cannot be ignored in the present study, the next appropriate step would be to determine an F-distributed variate in which the effect of sample size is removed. Furthermore, determining the statistic under which average $\text{tr}(E'E)$ is distributed would also be worthwhile.

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APPENDIX

Five variables - three components

Harman, H. Modern factor analysis, Chicago: University of Chicago Press, 1967, p. 137.

Hunka, S. An Investigation of Five Textbook Variables, Edmonton: University of Alberta, 1969.

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Six variables - three components

Morrison, D. F. Multivariate statistical methods, New York: McGraw-Hill Book Company, 1967, p. 243 and 255.

Eight variables - two components

Harman, H. Modern factor analysis. Chicago: University of Chicago Press, 1967. p. 164 and 166.

Ten variables - four components

Mosychuk, H. Differential home environments and mental ability patterns, Unpublished Doctoral Dissertation. University of Alberta, 1969.

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Eleven variables - six components

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Twelve variables - four components

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Twelve variables - six components

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Thirteen variables - three components

- Glass, G. V. and Maguire, T. O. Abuses of factor scores, American Educational Research Journal, 1966, 3, 297-304.

Thirteen variables - four components

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Sixteen variables - three components

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Eighteen variables - three components

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Eighteen variables - five components

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Twenty variables - six components

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